A Coded Modulation Scheme with Interblock Memory

Mao-Chao Lin, Member, IEEE, and Shang-Chib Ma, Student Member, IEEE

Abstract—In this paper, we propose a coded modulation scheme with interblock memory. The proposed scheme has a multi-stage coding structure. However, at some coding stage, the coding designs for every two adjacent blocks are related. Compared to coded modulation schemes for which each block is independent of the others, the introduction of interblock coding provides us with additional flexibility in the design. In this way, for the proposed scheme, we can achieve large minimum Euclidean distance while the number of nearest neighbors is not too large. We also design a multi-stage decoding algorithm, which is optimum in each stage and has low decoding complexity. Three specific examples are provided to demonstrate the advantages of the proposed scheme.

I. INTRODUCTION

EVER since Ungerboeck [1] presented the technique of using combined coding and modulation (coded modulation) to achieve appreciable coding gains without sacrificing the bandwidth, a lot of research results in this area have been reported [2]-[11]. One approach of designing coded modulation is based on a block by block manner. Such an approach usually has the multi-stage coding structure, which is also known as the Block-Zyablov coding structure [12]. In this paper, we call the coded modulation constructed from a block by block manner a block coded modulation (BCM) and call each encoded signal a coded signal block. We can easily construct BCM schemes with large minimum Euclidean distances (ED) by choosing proper binary block codes with proper minimum Hamming distances. If we want to achieve large minimum ED in a BCM scheme, usually long block length is required. However, the huge amount of neighboring coded signal blocks may seriously reduce coding gains at low or moderate signal to noise ratios. Furthermore, the formidable decoding complexities for long block codes prohibit the practical usage of BCM with long block design.

In this paper, we propose a coded modulation scheme which is a modified version of the BCM scheme proposed by Sayegh [10]. In the proposed scheme, the coded modulation of each block depends on the preceding block. The goal of such a BCM with interblock memory is to avoid the aforementioned constraints of conventional BCM while achieving a relatively large coding gain. The coding strategy for the proposed scheme is still based on the multi-stage coding structure. However, at some coding stage, the coding designs for every two adjacent blocks are related. Compared to BCM schemes for which each block is independent of the others, the introduction of interblock coding provides us with additional flexibility in the design. In this paper, the binary code used in the interblock coding has code length twice as large as the length of each block and is constructed by combining two short binary codes which was first introduced in [14]. In this way, for the proposed scheme, we can achieve large minimum ED while the number of nearest neighbors is not too large. Hence, substantial coding gains at moderate signal to noise ratios can be achieved. The structure of the binary code that is designed by combining two short codes enables us to design an efficient decoding algorithm for the corresponding coding stage which is similar to the squaring construction in [8] and is optimum in this stage. The decoding algorithm proposed in this paper is a two-stage decoding type and is optimum in each decoding stage.

We construct three specific examples for the proposed scheme. Example 1 is a coded modulation of block length 8 with 8-AMPS signal set geometry for which the asymptotic coding gain over uncoded QPSK is 4.77 dB. Simulated results show that, at the block error rate (BKER) of $10^{-6}$, Example 1 using the proposed decoding algorithm provides a coding gain of about 3.35 dB over uncoded QPSK of block length 8. We also provide theoretical analysis, which yields results close to the simulated results. The decoding complexity is low in consideration of the amount of coding gain. The price we have to pay is that Example 1 requires a small bandwidth expansion compared to uncoded QPSK. The bandwidth expansion factor is 16/15. Further theoretical analysis estimates that at the BKER of $10^{-9}$, the coding gain of Example 1 over uncoded QPSK of block length 8 will be around 3.68 dB. Example 2 is a coded modulation of block length 8 with 8-PSK signal set geometry for which the asymptotic coding gain over uncoded QPSK is 3.27 dB. Compared to uncoded QPSK, Example 2 has a bandwidth saving for which the ratio is 16/17. Comparisons of Example 1 and Example 2 with some other BCM schemes of block length 8 are also provided in this paper. Example 3 is a coded modulation of block length 15 with 8-PSK signal set geometry which has an asymptotic coding gain of 5.87 dB over uncoded QPSK. Theoretical estimation shows that at the BKER of $10^{-9}$, the coding gain of Example 3 over uncoded QPSK of block length 15 is 4.62 dB. Compared to uncoded QPSK, this...
example requires a bandwidth expansion for which the ratio is 30/29. Finally, we propose a method to reduce the bandwidth expansion factors required in Example 1 and Example 3.

II. THE PROPOSED CODED MODULATION SCHEME

Consider a class of binary linear codes, which was first constructed in [14]. Let $C$ be a $(2n, k_a + k_b + r)$ binary code with generator matrix of the following form:

$$G = \begin{bmatrix} G_{br} & G_{ar} \\ 0 & G_b \end{bmatrix},$$

where each 0 represents an all zero matrix, $G_a$ and $G_b$ are $k_a \times n$ and $k_b \times n$ matrices, respectively, $G_{ar}$ and $G_{br}$ are both $r \times n$ matrices. Let $C_a$, $C_b$, $C_{ar}$ and $C_{br}$ be the binary codes using $G_a$, $G_b$, $G_{ar}$ and $G_{br}$ as generator matrices, respectively. Let $C_{ab}$ and $C_{bb}$ be the binary codes using $G_a^T G_{ar}$ and $G_b^T G_{br}$ as generator matrices, respectively. Note that $C_{aa}$ is the direct sum of $C_a$ and $C_{ar}$. In notation, $C_{aa} = C_a \oplus C_{ar}$. Similarly, $C_{bb} = C_b \oplus C_{br}$. From the generator matrices, we see that $C_a$, $C_b$, $C_{aa}$, and $C_{bb}$ are $(n, k_a), (n, k_b), (n, k_a + r)$ and $(n, k_b + r)$ binary codes, respectively. Assume that the minimum Hamming distances of $C_a$, $C_b$, $C_{aa}$ and $C_{bb}$ are $d_{aa}$, $d_{bb}$, $d_{aa}$, and $d_{bb}$, respectively.

We may write each message $x$ of $C$ as the combination of three component messages, i.e., $x = (z_1, z_2, z_3)$, where $z_1$, $z_2$ and $z_3$ are $r$-tuple, $k_a$-tuple and $k_b$-tuple, respectively. Then, each codeword in $C$ can be encoded in the form, $(z_1, z_2, z_3 - G$.

Consider a coded modulation scheme based on 8-AMP signals as shown in Fig. 1(b). Let $n$ be the block length. Note that each signal symbol in the signal space can be represented by three bits, denoted $(a, b, c)$. Let $\vec{v} = (a_1, b_1, c_1, \cdots, a_n, b_n, c_n)$ and $\vec{v}' = (a_1', b_1', c_1', \cdots, a_n', b_n', c_n')$ represent two consecutive signal blocks. The combination of all adjacent blocks, represented by $(\vec{v}, \vec{v}')$, is called a superblock. In our scheme, $(c_1, \cdots, c_n)$ and $(c_1', \cdots, c_n')$ are codewords of an $(n, k)$ binary code with minimum Hamming distance $d_{bb}$. Moreover, $(b_1, \cdots, b_n, a_1', \cdots, a_n') = \vec{v} \cdot \vec{v}'$ is a codeword in $C$. The configuration of coding is shown in Fig. 2, which has a two-stage coding structure. Let $\vec{v} = (a_1, b_1, c_1, \cdots, a_n, b_n, c_n)$ and $\vec{v}' = (a_1', b_1', c_1', \cdots, a_n', b_n', c_n')$ be combined to represent another superblock. We may calculate the Euclidean distance between the coded signal superblocks represented by $(\vec{v}, \vec{v}')$ and $(\vec{v}, \vec{v}')$. Suppose that $(a_1, \cdots, a_n) = (a_1', \cdots, a_n)$. Let $(b_1, \cdots, b_n, a_1', \cdots, a_n') = \vec{v} \cdot \vec{v}'$, where $\vec{v} = (\vec{x}_1, \vec{x}_2, \vec{x}_3)$ and $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$ are $r$-tuple, $k_a$-tuple and $k_b$-tuple, respectively. Consider the following conditions.

(i) Suppose that $\vec{x}_1 \neq \vec{x}_1'$. Then, $D_1$, the minimum Euclidean distance (ED) between the coded signal superblocks represented by $(\vec{v}, \vec{v}')$ and $(\vec{v}, \vec{v}')$ satisfies

$$D_1^2 \geq 0.8 \cdot d_{aa} + 1.6 \cdot d_{bb}. \quad (2)$$

(ii) Suppose that $\vec{x}_1 = \vec{x}_1'$ and $\vec{x}_2 \neq \vec{x}_2'$. Then, $D_2$, the minimum ED between the coded signal blocks represented by $\vec{v}'$ and $\vec{v}'$ satisfies

$$D_2^2 = 0.8 \cdot d_{aa}. \quad (3)$$

(iii) Suppose that $\vec{x}_1 = \vec{x}_1'$, $\vec{x}_2 = \vec{x}_2'$ and $\vec{x}_3 \neq \vec{x}_3'$. The minimum ED, $D_3$, between the coded signal blocks represented by $\vec{v}$ and $\vec{v}'$ satisfies

$$D_3^2 = 1.6 \cdot d_{aa}. \quad (4)$$

(iv) Suppose that $(b_1, \cdots, b_n) = (b_1', \cdots, b_n')$. Since we assume $(a_1, \cdots, a_n) = (a_1', \cdots, a_n)$, the minimum ED, $D_4$, between the coded signal blocks represented by $\vec{v}$ and $\vec{v}'$ satisfies
With proper arrangement of rows in $G_{ar}$ and $G_{br}$, the inequality in (2) can be replaced by equality. If the condition of\[\min\{0.8 \cdot d_{aa} + 1.6 \cdot d_{bb}, 0.8 \cdot d_{aa}, 1.6 \cdot d_{bb}\} \geq 3.2 \cdot d_0\]is satisfied, then the square of minimum ED of the coded modulation scheme is $3.2 \cdot d_0$.

Example 1: Let $n = 8$. Let $C_a, C_{aa}, C_b, C_{bb}$ and $C_0$ be $(8,1,8), (8,4,4), (8,7,2)$ and $(8,7,2)$ binary extended BCH (or Reed-Muller) codes, respectively. The generator matrix of $C$ is shown in Fig. 3. Then, we have $d_{aa} = 4, d_{bb} = 2, d_a = 8, d_b = 2, r = 3, k_a = 1, k_b = 4$ and $k_0 = 7$. As a result, $D_1^2 = D_2^2 = D_3^2 = 6.4$. The code rate is 15/8 information bits per signal symbol. Compared to uncoded QPSK, this design has an asymptotic coding gain of 4.77 dB and requires a small bandwidth expansion for which the ratio is 16/15.

Example 2: Let $C_a, C_{aa}, C_b, C_{bb}$ and $C_0$ to be the $(8,1,8), (8,2,4), (8,7,2), (8,8,1)$ and $(8,8,1)$ binary codes, respectively, where the generator matrix of $C_{aa}$ is

$$G = \begin{bmatrix} G_{br} & G_{ar} \\ G_b & 0 \end{bmatrix}$$

where $G_{ar} = G_{bb} = 0$. The code rate of this scheme is 17/84 information bits per signal symbol. We have $D_1^2 = 4.344, D_2^2 = 4.688$ and $D_3^2 = 4$. Compared to uncoded QPSK, this design has an asymptotic coding gain of 3.27 dB and has a small bandwidth saving for which the ratio is 16/17.

Example 3: Let $C_a, C_{aa}, C_b, C_{bb}$ and $C_0$ to be the $(15,1,15), (15,5,7), (15,10,4), (15,14,2)$ and $(15,14,2)$ binary BCH codes, respectively. We have $D_1^2 = 8.10, D_2^2 = 8.79$ and $D_3^2 = 8$. The code rate of this scheme is 29/15 information bits per signal symbol. Compared to uncoded QPSK, this design has an asymptotic coding gain of 5.87 dB and requires a small bandwidth expansion for which the ratio is 30/29.

### III. Decoding Algorithm

We now propose a two-stage decoding algorithm. For each superblock, by assuming $(a_1, \ldots, a_n)$ is correct, we decode $(b_1, \ldots, b_n, a_1^*, \ldots, a_n^*)$ first and then decode $(c_1, \ldots, c_n)$. The decoding of $(b_1^*, \ldots, b_n^*)$ and $(c_1^*, \ldots, c_n^*)$ is processed in the decoding of the following superblock. Note that $C$ is a $(2n, k_b + k_a + r)$ binary linear code. We can apply the technique in [15] to obtain a code trellis and then achieve an optimum decoding of $(b_1, \ldots, b_n, a_1^*, \ldots, a_n^*)$. The trellis requires $2^r$ states, where $\ell = 2n - k_a - k_b - r$. For each of the three examples in this paper, the number of required states is too large. Therefore, we have to find an alternative design, which is similar to the squaring construction in [8].

Each codeword $u$ of $C_{bb} = C_b \oplus C_{bb}$ can be uniquely expressed as the sum of a codeword $u_b$ in $C_b$ and a codeword $u_{br}$ in $C_{br}$. We have $u = u_b + u_{br}$. Let $H_b$ be a parity check matrix of $C_b$. Then, $H \cdot H_b = H_{bb} \cdot H_b$. The set $S = \{u \cdot H_b : u \in C_{bb} = C_b \oplus C_{br}\}$ contains $2^r$ distinct elements. We may partition the $2^{k_b+r}$ codewords of $C_{bb}$ into $2^r$ disjoint sets. That means

$$C_{bb} = C_b \oplus C_{br} = \bigcup_{u_{br} \in C_{br}} \{u_b + u_{br} : u_b \in C_b\},$$

where $u_{br} \in C_{br} = \{u_{br} + u_b : u_b \in C_b\}$. For all the $u$ in a given $u_{br} \in C_{br}$, the values of $u \cdot H_b$ are the same.

We may construct a $2^{n-k_b}$-state trellis for $C_b$ which is a slightly modified version of the trellis shown in [15]. In [15], each state of the trellis is represented by an $(n - k_b)$-tuple and each path starts from the state represented by the all zero vector at time zero and is back to the same state at time $n$. In our modified trellis, we have $2^r$ distinct remaining states at time $n$, which are in fact represented by the $2^r$ elements of $S$. For code $C_b$ of Example 1, its parity check matrix $H_b$ and the modified trellis are shown in Figs. 4 and 5, respectively. By applying Viterbi algorithm on the modified trellis, we have $2^r$ survivor paths with associated metrics for the $2^r$ states at time $n$. The survivor path for the state represented by $(u_{br} + C_{bb}) \cdot H_b$ is the path among the $2^{2^r}$ paths represented by codewords in $u_{br} \oplus C_b$ which is closest to the received path and the associated metric is the square of ED between the survivor path and the received path. This survivor path may be called the best path of $u_{br} \oplus C_b$. We can similarly decode $C_{aa} = C_a \oplus C_{ar}$ using $H_a$. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ and denote the
2n-tuple, \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) by \((\overline{x}, \overline{y})\). Each codeword of \(C\) is of the form \((\overline{u}_a + \overline{u}_{ar}, \overline{u}_a + \overline{u}_{ar})\), where \(\overline{u}_a \in C_a\), \(\overline{u}_{ar} \in C_{ar}\), \(\overline{u}_b \in C_b\), \(\overline{u}_{br} \in C_{br}\). Note that for a given codeword \(\overline{u}_{ar} \in C_{br}\), the corresponding codeword \(\overline{u}_{ar} \in C_{ar}\) is determined. Therefore, for a given \(\overline{u}_{ar} \in C_{br}\) together with the corresponding \(\overline{u}_{ar} \in C_{ar}\), we can combine the best path of \(\overline{u}_{br} \in C_{br}\) with the best path of \(\overline{u}_{ar} \in C_{ar}\) to be the best path of all the codewords of the set \(T(\overline{u}_{br}, \overline{u}_{ar}) = \{(\overline{u}_b + \overline{u}_{br}, \overline{u}_a + \overline{u}_{ar}) : \overline{u}_b \in C_b, \overline{u}_a \in C_a\}\) and determine the associated metric. There are \(2^n\) distinct \(T(\overline{u}_{br}, \overline{u}_{ar})\). From the \(2^n\) best paths and the associated metrics of the \(2^n\) distinct \(T(\overline{u}_{br}, \overline{u}_{ar})\), we can determine the best path among all the paths represented by codewords of \(C\) and determine the associated metric. The decoding of \((c_1, \ldots, c_n)\) can be done by using the code trellis of \(C_0\) constructed by applying the technique of [15]. For the special case such as Examples 1, 2 or 3, the number of codewords in \(C_{pa}\) is small, we can directly calculate ED between the received path and all the paths represented by codewords of \(C_{pa}\) to determine the best path instead of resorting to the code trellis.

![Parity check matrix of \(C_0\) in Example 1.](image)

![Trellis of \(C_0\) in Example 1.](image)

Note that in this two-stage decoding procedure, we can find the best path among all the paths represented by codewords of \(C\) and the best path among all the paths represented by codewords of \(C_0\). Hence, the proposed decoding procedure is optimum in each stage.

IV. Performance Analysis

Consider Example 1. Using the decoding algorithm proposed in Section III, we derive that the total number of required branch additions for decoding one block (8 signal symbols) is 306 and the total number of required comparisons is 84. In average, we need 38.25 additions and 10.5 comparisons per signal symbol interval. With this decoding algorithm, we obtain simulated results which are shown in Fig. 6, where \(E_b\) is the required energy per information bit and \(N_0\) is the one-sided power spectral density. We may compare simulated results of Example 1 to simulated or theoretical results of other BCM schemes of block length 8 which are also shown in Fig. 6.

Case 1: This is the uncoded QPSK of block length 8. At the block error rate (BKER) of \(10^{-6}\), the coding gain of Example 1 over Case 1 is about 3.35 dB. Compared to Case 1, Example 1 requires a small bandwidth expansion for which the ratio is 16/15.

![Coding performance. (a) Uncoded QPSK. (b) Block coded modulation in [13]. (c) Simulation results of Case 3. (d) Simulation results of Case 4. (e) Simulation results of Case 5. (f) Simulation results of Example 1.](image)
17/8 information bits per signal symbol. Compared to uncoded QPSK, Case 3 has an asymptotic coding gain of 3.27 dB and has a bandwidth saving for which the ratio is 16/17. Using the decoding algorithm proposed in Section III, the numbers of required additions and comparisons per signal symbol are 8 and 3.13, respectively. At the BKER of 10^{-6}, the coding gain of Example 1 over Case 3 is around 0.80 dB. Compared to Case 2, we see that Case 3 requires a similar decoding complexity, a smaller bandwidth and has higher coding gains.

Case 4: This is a BCM scheme of block length 8 which is designed based on the 8-AMPM signal set geometry and has a three-stage coding structure. The binary codes for the three stages are (8,4,4), (8,7,2) and (8,8,1) codes, respectively. For this scheme, the code rate is 19/8 information bits per signal symbol, the minimum ED is $\sqrt{3.2}$. Compared to uncoded QPSK, the asymptotic coding gain is 2.79 dB and there is some bandwidth saving for which the ratio is 16/19. Using a three-stage decoding algorithm which is optimum in each stage, the numbers of required additions and comparisons per signal symbol are 9.5 and 4, respectively. At the BKER of 10^{-6}, the coding gain of Example 1 over Case 4 is around 1.35 dB.

Case 5: This is a BCM scheme of block length 8 which is designed based on the 8-AMPM signal set geometry and has a three-stage coding structure. The binary codes for the three stages are (8,1,8), (8,4,4) and (8,7,2) codes, respectively. For this scheme, the code rate is 1.5 information bits per signal symbol, the minimum ED is $\sqrt{6.4}$. Compared to uncoded QPSK, the asymptotic coding gain is 3.80 dB, while a bandwidth expansion for which the ratio is 4/3 is required. Using a three-stage decoding algorithm which is optimum in each stage, the numbers of required additions and comparisons per signal symbol are 11.5 and 3.12, respectively. At the BKER of 10^{-6}, the coding gain of Example 1 over Case 5 is around 0.4 dB.

We may compare simulated results of Example 1 to data obtained from theoretical analysis. According to [1], the probability of incorrect decoding can be roughly estimated by

$$Pr(e) \approx N \cdot Q(D_{min}/2\sigma),$$

where $N$ is the number of neighbors at a distance $D_{min}$ (minimum ED of the coded modulation scheme), $\sigma$ is the standard deviation of the additive white Gaussian noise and $Q(y)$ is the Gaussian error probability function. For uncoded QPSK of block length 8, the number of nearest neighbors, $N$, is 16 and the minimum ED, $D_{min}$, is $\sqrt{2}$. From (11), we find that for $Pr(e) = 10^{-6}, 1/(\sqrt{2}\sigma)$ is roughly equal to 5.29, which indicates that the required $E_s$ to $N_0$ ratio is 14.45 dB, or equivalently the required $E_b$ to $N_0$ ratio is 11.44 dB, where $E_b$ is the required energy per signal symbol in the signal space. For a two-stage decoding, we have

$$Pr(e) \approx N_1Q(D^{(1)}/2\sigma) + N_2Q(D^{(2)}/2\sigma),$$

where $N_1$ and $N_2$ are the numbers of nearest neighbors at stages 1 and 2 of decoding, respectively. $D^{(1)}$ and $D^{(2)}$ are the minimum ED at stages 1 and 2 of decoding, respectively. In Example 1, $D^{(1)} = D^{(2)} = \sqrt{6.4}$, while $N_1$ is about 6622 and $N_2$ is 28 on statistical average. Using (12), we find that for $Pr(e) = 10^{-6}$, the required value of $1/(\sqrt{2}\sigma)$ is 3.52, which indicates an $E_b$ to $N_0$ ratio of 8.20 dB, which is close to the simulated result. We may estimate the performance of Example 1 at the BKER of $10^{-9}$. For $Pr(e) = 10^{-9}$, the uncoded QPSK of block length 8 and Example 1 require $E_b$ to $N_0$ ratios of 13.16 dB and 9.48 dB, respectively. Hence, at the BKER of $10^{-9}$, Example 1 has a coding gain of 3.68 dB over uncoded QPSK of block length 8.

Using similar analysis, we estimate that, for Example 3, at the BKER of $10^{-9}$, the coding gain over uncoded QPSK of block length 15 is 4.62 dB, where the contribution of errors from the second, the third and the fourth nearest neighbors are taken into account. The decoding complexity is about two times of that for Example 1.

V. FURTHER MODIFICATION

We can significantly reduce the required bandwidth expansion for Examples 1 and 3 by slightly modifying the aforementioned coding scheme. Note that the number of nearest neighbors, $N_2$, at the decoding of stage 2 is very small compared to $N_1$. Hence, the contribution of decoding errors at stage 2 is negligible if the decoding at stage 1 is correct. We are now intended to modify the encoding and decoding for stage 2. In the modified scheme, the encoding of $(a_1, \ldots, a_n), (a'_1, \ldots, a'_n), (b_1, \ldots, b_n)$ and $(b'_1, \ldots, b'_n)$ are the same as in Section II. Let $\bar{u}, \bar{v}, \bar{v}^{**}, \bar{v}^{***}, \ldots$, represent a sequence of signal blocks. We combine $(c_1, \ldots, c_n)$ and $(c'_1, \ldots, c'_n)$ as a codeword of a $(2n, 2n - 1, 2)$ even parity binary code, $C_0$. The combination of $(c_1^{**}, \ldots, c_n^{**}), (c'_1^{**}, \ldots, c'_n^{**})$ is also a codeword of $C_0$. In this way, each time we encode $2(k_a + k_b + r) + 2n - 1$ information bits into a big block of $2n$ signal symbols. The configuration of coding is shown in Fig. 7.

**Fig. 7.** Configuration of modified coding in Section V.

Here, we call the combination of two blocks a big block which is different from a superblock that is defined earlier. Note that two consecutive superblocks have overlap of one block while two consecutive big blocks do not overlap. For such a modification scheme, the decoding at stage 1 is exactly the same as that in Section III, and the decoding at stage 2 is simply changing the two-state trellis of $C_0$ into the two-state trellis of $C'_0$. Hence, the decoding complexity of the modified
scheme is the same as the original scheme. Consider the error rate of big blocks for the modified scheme. The number of nearest neighbors, $N_2$, at stage 2 of decoding for the modified scheme is about four times of $N_2$ of the original scheme. Usually, $N_2$ is still small compared to $N_1$. Hence, the contribution of errors is mostly from the decoding errors at stage 1. Since a big block contains two blocks, we may roughly predict that the error rate of big blocks for the modified scheme will be about two times of the error rate of blocks for the original scheme under the same signal to noise ratio. Note that, for uncoded QPSK, the error rate of big blocks (of $2n$ signal symbols) is also two times of the error rate of blocks (of $n$ signal symbols).

Consider some numerical data. For the modification of Example 1, we encode 31 information bits into a big block of 16 signal symbols each time. The code rate is $31/16$ information bits per signal symbol and hence the required bandwidth expansion factor compared to uncoded QPSK is $32/31$, equivalently to a $0.15$ dB reduction of $E_b/N_0$ ratios. Note that in Example 1 the reduction of $E_b$ to $N_0$ ratios due to the bandwidth expansion is $0.28$ dB. The number of nearest neighbors, $N_2$, at stage 2 of decoding is 120, which is still very small compared to $N_1 = 6622$. Through simulation, we find that, at the error rate of big blocks of $10^{-4}$, the modified scheme of Example 1 has a coding gain of about $3.3$ dB over uncoded QPSK of big block length 16. The simulation result is close to our theoretical analysis which estimates an additional coding gain of $0.13$ dB because of the smaller bandwidth expansion. It is then clear that the modified scheme described in this section is even superior to the original scheme described in Section II.

ACKNOWLEDGEMENT

The authors would like to thank the referees and the associate editor for their valuable comments which greatly improve the quality of this paper.

REFERENCES